

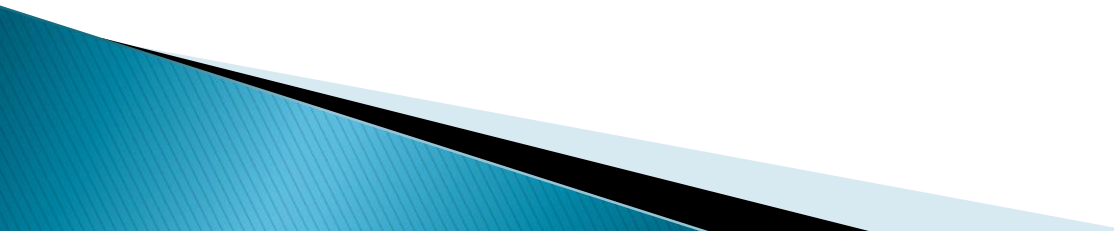
Warm-up Puzzle

- ▶ One hundred prisoners are lined up facing the same direction, single file, so that each prisoner can see the people in front of him but no one behind him. A red or black hat is placed on each person's head, but no one can see the color of his/her hat. They can, however, see the hats of the people in front.
- ▶ Their jailer tells them they will each be given an opportunity to guess the color of their hat, starting from the last person in line. Those who guess correctly will walk free. Everyone can hear the guesses.
- ▶ The prisoners are allowed to discuss a strategy before lining up. How can they maximize the number of people who will be released?

Counting and Probability

Mostly counting today.

Some definitions

- ▶ Probability is defined as the number of ways something can happen divided by the total number of things that can happen.
 - ▶ To find out each of these numbers, we have to learn how to count!
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Permutations and Factorials

- ▶ A permutation of a collection of distinguishable objects is an arrangement of the objects in some specific order.
- ▶ The number of distinct permutations of n objects is $n(n - 1)(n - 2) \dots (2)(1) = n!$
- ▶ The “!” symbol is called a factorial.

Permutation problem

- ▶ Using the letters A, M, O, S, and U, we can form $5! = 120$ five-letter “words.” If these “words” are arranged in alphabetical order, then what position does the “word” USAMO occupy?

Combinations

- ▶ Basic counting problems involve the number of ways of choosing items from a set.
- ▶ How many ways are there to choose 2 people from 4?
- ▶ Number of ways to choose the first person: 4
- ▶ Number of ways to choose the second person: 3 (one of the people has already been chosen)
- ▶ Did we double count? Rearrange the two people in $2!=2$ ways, each of which is equivalent.
- ▶ Total:
$$\frac{4 \times 3}{2} = 6$$

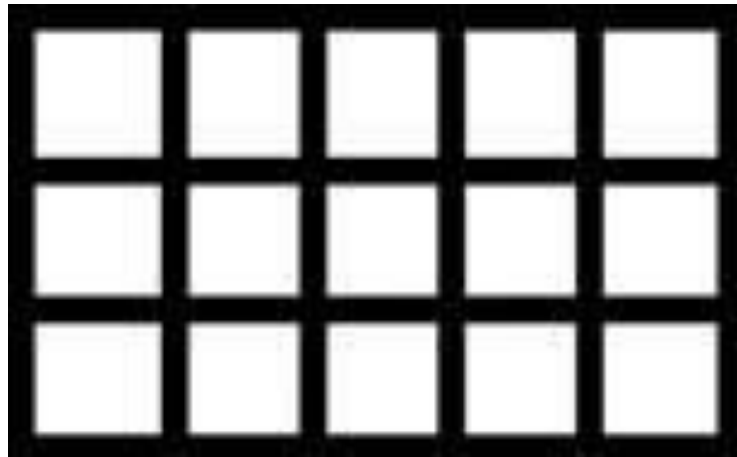
Combinations: Definition

- ▶ The number of ways of choosing k objects from a collection of n distinguishable objects, where the order of choosing does not matter, is

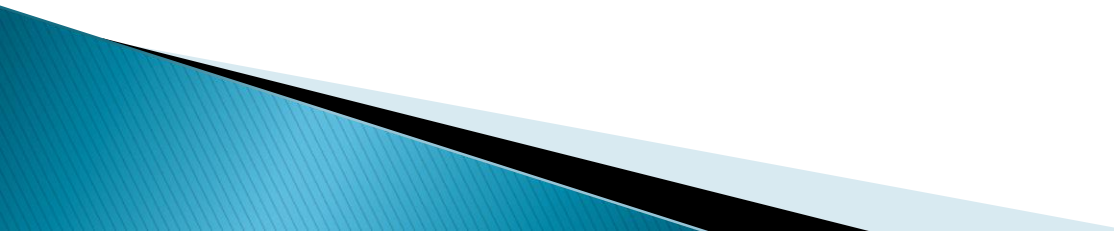
$$\binom{n}{k} = \frac{n!}{k! (n - k)!}$$

Practice Counting

- ▶ How many ways are there to move from the bottom left corner of the grid to the top right corner if you can only move up or right one unit at a time?



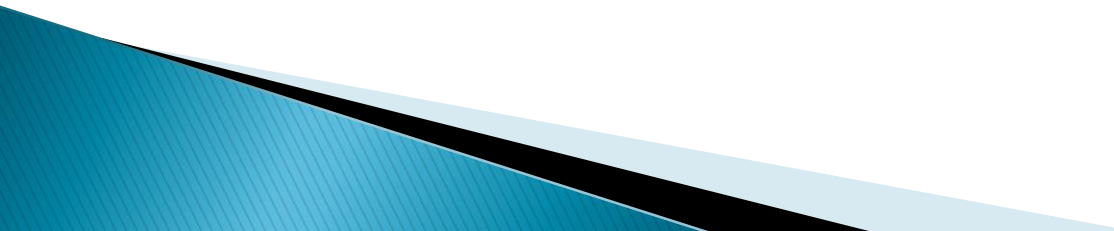
Casework

- ▶ Casework is the most basic of counting methods. It involves listing distinct cases and counting for each case separately.
 - ▶ Though casework often presents a viable solution, too many cases indicate an easier solution.
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A problem.

- ▶ Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

Complementary Counting

- ▶ Sometimes it's easier to count what we don't want instead of counting what we do.
 - ▶ Subtract that number from the total and we get our desired result.
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No one

- ▶ How many numbers from 1 to 1000, inclusive, do not contain the number 1?

Stars and Bars

- ▶ Let's look at the same problem we just solved.
- ▶ Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

Which ones are the stars?

- ▶ Since we are choosing six cookies, let's represent each cookie with a star.



- ▶ Each cookie is either chocolate chip, oatmeal, or peanut butter. So there are three groups of cookies if we separate them by type.
- ▶ Let's add dividers to indicate three groups.



More Stars and Bars

- ▶ So each combination of stars and bars represents a case.
- ▶ For example,



indicates 6 of the first type and none of the next two.

- ▶ Having 3 of the first type, 2 of the second, and one of the third looks like this:



So what?

- ▶ This means we can use combinations to solve the problem!
- ▶ We have eight items and we want to choose two of them to be our bars.

$$\frac{8!}{2!(8-2)!} = 28$$

- ▶ In general, if we are choosing n objects with k possible types, we can find

$$\binom{n+k-1}{k-1}$$

Practice with Stars and Bars

- ▶ Pat wants to buy five donuts from an ample supply of three types of donuts: glazed, chocolate, and powdered. How many different selections are possible?

To challenge your thinking

- ▶ A spider has one sock and one shoe for each of its eight legs. On each leg the sock must be put on before the shoe. In how many different orders can the spider put on its socks and shoes?