

Inequalities

Newport Math Club

Warm up question:

You are presented with five cards shown as follows:



Given that each card has one letter on one side and one number on the other side, which cards need to be flipped in order to determine the validity of the following statement:

“Each card with a K on one side, has a 3 on the other side.”

What is an inequality?

- Definition:

A statement relating two expressions.

- Examples:

- $A > B$

- $A < B$

- $A \neq B$

- $A \leq B$

- $A \geq B$

Basic inequality problem

Suppose a student receives the following scores on his calc test: 85, 81, 99, 83, 92. What must he score on the next test to average at least 89.5 in the class, so his teacher will round up his grade?

The Triangle Inequality

(related, but not really related)

- In any “normal” triangle, the sum of any two sides must **be greater** than the third.
- This is a strict inequality meaning the sum of two sides **cannot equal** the third side.

The Triangle Inequality (cont.)

- A triangle that does not satisfy the triangle inequality is called degenerate. Here are some examples



- What is the area of a degenerate triangle?
 - 0 or \emptyset (the null set), but not both.

The Triangle Inequality (cont.)

The triangle inequality is often written as

$$|z| + |w| \geq |z + w|$$

Where z and w are complex numbers.

What's the connection between complex numbers and triangles?

The Trivial Inequality

- Definition: For all real numbers X , the trivial inequality states: $X^2 \geq 0$, with equality iff $X = 0$.
- Why is this true?
 - The squares of positive numbers are positive
 - The squares of negative numbers are positive
 - The square of 0 is 0.
- Why do we care?
 - It's one of the most commonly used inequalities in mathematics.

Problem

Prove that $x^2 + 2x + 1 \geq 0$ for real x .

Problem

Prove that $4x^2 + 9y^2 \geq 12xy$ for all real x, y

$$\begin{aligned} & 4x^2 - 12xy + 9y^2 \\ &= (2x - 3y)^2 \geq 0 \end{aligned}$$

Problem

Prove that $\cos(2\theta) - \cos^2(\theta) + 1$ is nonnegative for all real angles θ .

Problem

Prove: $(x^2 + 1)(y^2 + 1) \geq (xy + 1)^2$ for all real x, y

Problem

Find the minimum value of $x^2 + 2x + 2$ using the trivial inequality. Do not use $-\frac{b}{2a}$.

AM - GM

- The AM – GM inequality states that for positive n and positive $a_1, a_2, a_3, \dots, a_n$ we have:

$$\frac{a_1 + a_2 + a_3 + \dots + a_n}{n} \geq \sqrt[n]{a_1 a_2 a_3 \dots a_n}$$

- Looking at the expressions, can you guess what AM – GM stands for?

Arithmetic mean – Geometric mean inequality

Example

- Consider the numbers 2, 4, and 8.
- The arithmetic mean is just $\frac{2 + 4 + 8}{3} = \frac{14}{3}$.
- The geometric mean is $\sqrt[3]{(2)(4)(8)} = \sqrt[3]{4^3} = 4$.
- And $\frac{14}{3} > 4$.
- Coincidence? Nah.

Problem

The general AM – GM inequality is somewhat tricky to prove. However the case $n = 2$ is quite easy.

Prove:
$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}$$

Problem

Often times when we work with inequalities, we are interested in the equality case. That is, what conditions do $a_1, a_2, a_3, \dots, a_n$ have to satisfy to produce equality?

Find the equality case for:

$$\frac{a_1 + a_2}{2} = \sqrt{a_1 a_2}$$

Problem

Show that if $a_1 a_2 a_3 \dots a_n = 1$

and all terms are positive,

then $a_1 + a_2 + a_3 + \dots + a_n \geq n$

Problem

For positive x , find the minimum value of the function:

$$f(x) = x + \frac{9}{x}$$

hint: AM-GM

(if you use calculus on this, I will be disappointed)

Problem

- Find the minimum value of $\frac{9x^2 \sin^2(x) + 4}{x \sin(x)}$ for $0 < x < \pi$. Show this value can be obtained.

Problem

Prove that for all positive numbers x , y , and z :

$$x^2 + y^2 + z^2 \geq xy + yz + xz$$

Extensions of AM – GM

- It should be noted that the inequalities can be extended to:

HM – Arithmetic mean – Geometric mean – QM – PM

$$(HM \leq AM \leq GM \leq QM \leq PM)$$

- Can you guess what these stand for?

- Harmonic mean
$$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots + \frac{1}{a_n}}$$

- Quadratic mean
$$\sqrt{\frac{a_1^2 + a_2^2 + a_3^2 + \dots + a_n^2}{n}}$$

- ~~Power~~ mean
$$\sqrt[p]{\frac{a_1^p + a_2^p + a_3^p + \dots + a_n^p}{n}}$$

Note: These means are often not useful and are tested for the sake of testing definitions. So you should know them, but not really.