$(\mathbf{1a})$

Let

$$z = a + bi$$

$$w = c + di$$

LHS:

$$\overline{z + w}$$

$$\begin{aligned} &= \overline{(a + bi) + (c + di)} \\ &= \overline{(a + c) + (b + d)i} \\ &= (a + c) - (b + d)i \end{aligned}$$

RHS:

$$\overline{z} + \overline{w}$$

$$\begin{aligned} &= (\overline{a + bi}) + (\overline{c + di}) \\ &= (a - bi) + (c - di) \\ &= (a + c) - (b + d)i \end{aligned}$$

$(\mathbf{1b})$

LHS:

$$\overline{zw}$$

$$\begin{aligned} &= \overline{(a + bi)(c + di)} \\ &= \overline{ac + adi + bci + bdi^2} \\ &= (ac - bd) - (ad + bc)i \end{aligned}$$

RHS:

$$\overline{z}\,\overline{w}$$

$$\begin{aligned} &= \left(\overline{a + bi}\right)(\overline{c + di}) \\ &= (a - bi)(c - di) \\ &= ac - adi - bci + bdi^2 \\ &= (ac - bd) - (ad + bc)i \end{aligned}$$

$(\mathbf{1}\boldsymbol{c})$

LHS:

$$\overline{(z/w)}$$

$$= \overline{\left(\frac{a+bi}{c+di}\right)}$$

$$= \overline{\left(\frac{ac+bd}{c^2+d^2}\right)+\left(\frac{bc-ad}{c^2+d^2}\right)i}$$

$$= \left(\frac{ac+bd}{c^2+d^2}\right)-\left(\frac{bc-ad}{c^2+d^2}\right)i$$

RHS:

$$\overline{z}/\overline{w}$$

$$= \left(\overline{a+bi}\right)/\left(\overline{c+di}\right)$$

$$= \frac{a-bi}{c-di}$$

$$= \left(\frac{ac+bd}{c^2+d^2}\right)-\left(\frac{bc-ad}{c^2+d^2}\right)i$$

$(\mathbf{2})$

The sum $\frac{1}{r^2}+\frac{1}{s^2}+\frac{1}{t^2}$ can be expressed as

$$\frac{s^2t^2+r^2t^2+r^2s^2}{r^2s^2t^2}$$

Here's the tricky part, we try to rewrite the numerator and express the fraction as

$$= \frac{(st+rt+rs)^2-2(r^2st+rs^2t+rst^2)}{(rst)^2}$$

$$= \frac{(st+rt+rs)^2-2(rst)(r+s+t)}{(rst)^2}$$

From the polynomial $x^3-6x^2+5x-7=0$ we can find the following with Vieta's formulas

$$\begin{array}{rcl} r+s+t & = & 6 \\ rs+st+tr & = & -5 \\ rst & = & 7 \end{array}$$

Now the desired sum $\frac{1}{r^2}+\frac{1}{s^2}+\frac{1}{t^2}$ is

$$\begin{array}{rc} = & \frac{(-5)^2-2(7)(6)}{7^2} \\ = & -\frac{59}{49} \end{array}$$

Our answer is just $\boxed{-\frac{59}{49}}$

(3)

The roots are just the 8 8th of unity which form an octagon in the complex plane. The roots with positive part are:

$$\boxed{\sqrt{2}+\sqrt{2}i,\ \ 1\ ,\ \ \sqrt{2}-\sqrt{2}i}$$

(4)

The polynomial $x^4+x^2+1=0$ can be rewritten as

$$\frac{x^6-1}{x^2-1}=0$$

(recall the formula for the sum of a geometric series)

The zeros of x^6-1 is just the 6 6th roots of unity. But we cannot have 6 roots since the original polynomial is of degree 4. Why do we have 2 extra roots? It's because the denominator is 0 at $x=1$ and $x=-1$. Our desired roots are just:

$$\boxed{\frac{1}{2}+\frac{\sqrt{3}}{2}i,\quad \frac{1}{2}-\frac{\sqrt{3}}{2}i,\quad -\frac{1}{2}+\frac{\sqrt{3}}{2}i,\quad -\frac{1}{2}-\frac{\sqrt{3}}{2}i}$$

$(\mathbf{5})$

We are given that $(a + bi)z$ is equidistant from the origin and z This translates to

$$\begin{aligned} |(a+bi)z - z| &= |(a+bi)z| \\ |z(a-1) + bzi| &= |az + bzi| \\ |z||(a-1) + bi| &= |z||a+bi| \\ (a-1)^2 + b^2 &= a^2 + b^2 \end{aligned}$$

This gives

$$a = \frac{1}{2}$$

And it follows that

$$\boxed{b = \frac{255}{4}}$$

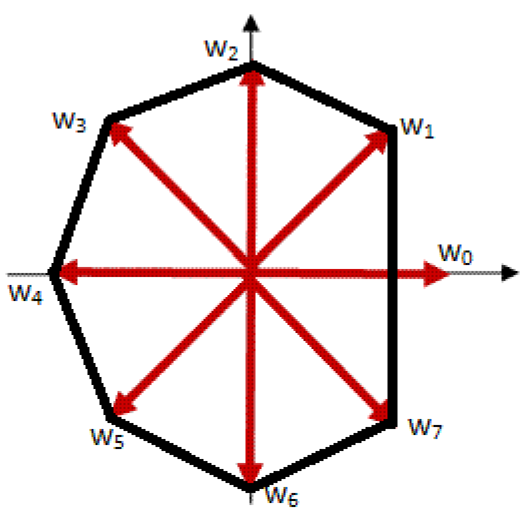
$(\boldsymbol{c1})$

First we rewrite the sum as

$$\frac{x^8 - 1}{x - 1} = 0$$

This is just the 8 8th roots of unity, but $x = 1$.

A sketch of the region



The area can be computed with simple geometry to get $\boxed{A = \frac{2+\sqrt{2}}{4}}$

$(c2)$

Suppose the four roots of $f(x) = x^4 - 3x^2 + x - 9 = 0$ are x_1, x_2, x_3, and x_4.

Their reciprocals are $\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}$, and $\frac{1}{x_4}$.

These reciprocals satisfy $\frac{1}{x^4} - \frac{3}{x^2} + \frac{1}{x} - 9 = 0$

But this is not a polynomial, so we multiply by x^4

$$g(x) = x^4 f\left(\frac{1}{x}\right) = 1 - 3x + x^3 - 9x^4$$

$(c3)$

We know from Euler's formula that

$$e^{ix} = \cos(x) + i\sin(x)$$

Suppose we take the imaginary part of both sides, then

$$Im(e^{ix}) = Im(\cos(x) + i\sin(x)) = \sin(x)$$

We can rewrite the integral as

$$\begin{aligned} &\int xe^x Im(e^{ix})dx \\ = \quad &Im\left(\int xe^x e^{ix} dx\right) \\ = \quad &Im\left(\int xe^{(1+i)x} dx\right) \end{aligned}$$

Integration by parts…

x	$e^{(1+i)x}$
1	$\frac{e^{(1+i)x}}{1+i}$
0	$\frac{e^{(1+i)x}}{(1+i)^2}$

Yields

$$= \quad Im\left(\frac{xe^{(1+i)x}}{1+i} - \frac{e^{(1+i)x}}{(1+i)^2}\right)$$

$$= \quad Im\left(\frac{xe^{x}e^{ix}(1-i)}{2} - \frac{e^{x}e^{ix}}{2i}\right)$$

$$= \quad Im\left(\frac{xe^{x}e^{ix}(1-i)}{2} + \frac{ie^{x}e^{ix}}{2}\right)$$

$$= \quad Im\left(\frac{xe^{x}(\cos(x) + i\sin(x))(1-i)}{2} + \frac{ie^{x}(\cos(x) + i\sin(x))}{2}\right)$$

$$= \quad Im\left(\frac{xe^{x}}{2}(\cos(x) - i\cos(x) + i\sin(x) + \sin(x)) + \frac{e^{x}}{2}(i\cos(x) - \sin(x))\right)$$

$$= \quad \frac{e^{x}}{2}(x\sin(x) - x\cos(x) + \cos(\mathrm{x})) + \mathrm{c}$$

$$\boxed{\frac{e^{x}}{2}(x\sin(x) - x\cos(x) + \cos(\mathrm{x})) + \mathrm{c}}$$