Problem Set 1
Complex Numbers and Polynomials

(1)
Prove the following properties of complex numbers:

$$\begin{array}{lccc} a. & \overline{z+w} & = & \bar{z}+\bar{w} \\ b. & \overline{zw} & = & \bar{z}\cdot\bar{w} \\ c. & \overline{(z/w)} & = & \bar{z}/\bar{w} \end{array}$$

(2)
Let r, s, and t be the roots of $x^3 - 6x^2 + 5x - 7 = 0$. Find

$$\frac{1}{r^2}+\frac{1}{s^2}+\frac{1}{t^2}$$

Hint: Remember Vieta's…

(3)
List the roots of $x^8 - 1 = 0$ with positive real part.

(4)
List the roots of $x^4 + x^2 + 1 = 0$

(5)
A function $\boldsymbol{f}$ is defined on the complex numbers by $\boldsymbol{f}(\boldsymbol{z}) = (\boldsymbol{a} + \boldsymbol{bi})\boldsymbol{z}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are positive numbers. This function has the property that the image of each point in the complex plane is equidistant from that point and the origin. Given that $|\boldsymbol{a} + \boldsymbol{bi}| = \mathbf{8}$ and that $\boldsymbol{b}^2 = \boldsymbol{m}/\boldsymbol{n}$, where $\boldsymbol{m}$ and $\boldsymbol{n}$ are relatively prime positive integers. Find $\boldsymbol{m} + \boldsymbol{n}$.

Challenge questions

(1)
Compute the area of the region defined by the roots of the polynomial $x^7 + x^6 + x^5 + \cdots + 1 = 0$ in the complex plane.

(2)
Find the polynomial whose roots are the reciprocals of the roots of $x^4 - 3x^2 + x - 9 = 0$.

Hint: If you're trying to find the actual roots, you're doin' it wrong.

(3)
For those that are taking Calculus, find the integral $\int xe^x \sin(x)\,dx$.

Hint: Remember Euler's formula … also, you might want to look up integration by parts.

(4)
An unfair coin with probability $\frac{1}{3}$ of heads and probability $\frac{2}{3}$ of tails is flipped 50 times. Find the probability, in closed form, of obtaining an even number of heads.