Polynomials Review and Complex numbers

Newport Math Club

Warm Up Problem:

The game of assassin is played with 8 people and goes as follows:

1. Each player is somehow assigned a target.
2. You attempt to assassinate your target.
   - if you succeed, you inherit the current target of the person you just killed
   - If you fail, you are removed from the game
3. The game ends when there is one player remaining.

Say there is no impartial player, the problem is to come up with a way to assign an 8 – cycle of targets and a system of communicating the inherited target to a killer.

Review 1

Find the quotient:

$$\frac{x^4 - 4x^3 + 5x^2 - 8x + 6}{x - 3}$$

Review 2

F(x) leaves a remainder of - 8 when divided by x + 3.

Find F(- 3).

Can I find F(3)?

Review 3

G(x) leaves a remainder of $2x - 1$ when divided by $x + 6$.

Find G(-6).

Review 4

Say r and s are the roots of the polynomial $h(x) = x^2 - 5x + 9$. Find the sum $\frac{1}{r} + \frac{1}{s}$.

Review 5

P(x) is a polynomial with real coefficients. When P(x) is divided by x – 1, the remainder is 3. When P(x) is divided by x – 2, the remainder is 5. Find the remainder when P(x) is divided by $x^2 - 3x + 2$.

hint: write P(x) as q(x)h(x) + r(x) where h is what you're dividing by and r is the remainder

What is a complex number?

- A number in the form of z = a + bi with real a, b and $i = \sqrt{-1}$

  (Note: if $b \neq 0$ then z is complex. In addition, if a = 0, then z is “purely imaginary”)

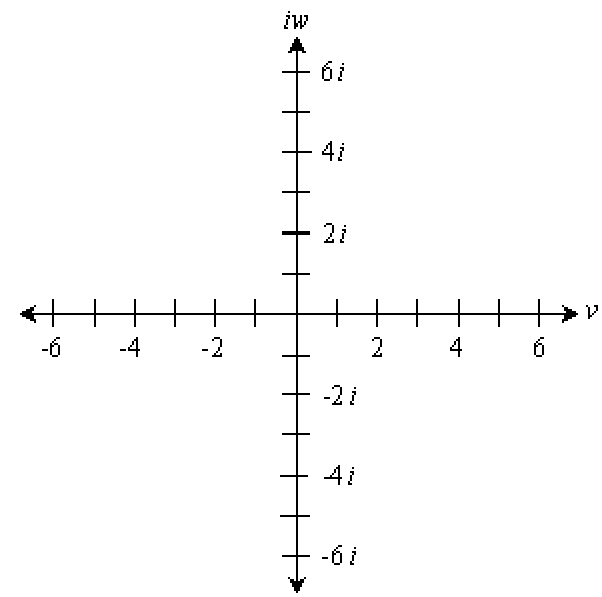
- Consider the equation: $x^2 + 1 = 0$

  Clearly, there are no real solutions, this is why we have complex numbers.

What is a complex number? (cont.)

- We represent numbers in the complex plane with the x axis representing the real part of the number, and the y axis representing the imaginary part of the number.

- The absolute value of a complex number, i.e., $|a+bi|$ is simply just the distance from the origin. This distance is $|a+bi| = \sqrt{a^2+b^2}$. This comes from the Pythagorean Theorem.



Basics of complex numbers

- The conjugate of a complex number $z = a + bi$ is defined as $a – bi$. Simply flip the sign on the imaginary part of the complex number.
- Why do we care? The conjugate is used in simplifying quotients involving complex numbers (along with a variety of other uses)

Question

Express $\frac{5+4i}{2-3i}$ as a single complex number.

Question

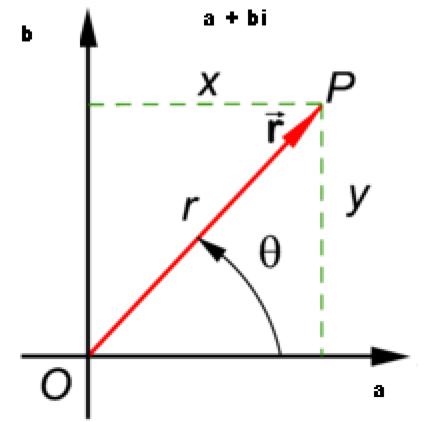
Suppose $z = a + bi$ is a complex number.

Real-ize the denominator of $\frac{1}{z}$.

Problem

Prove the following $\overline{zw} = (\overline{z})(\overline{w})$

Basics of complex numbers



It is often useful to write complex numbers in their polar representation. The polar representation of a point is expressed as $(r\ ,\ \theta)$.

where r is the distance from the origin and theta is the angle from the positive x axis. Given a and b, we know $r = \sqrt{a^2 + b^2}$. What is theta expressed in terms of a and b?

Basics of complex numbers

$$\theta = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0 \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ 0 & \text{if } x = 0 \text{ and } y = 0 \end{cases}$$

Multiplication of complex numbers

When multiplying complex numbers, we treat the i as a variable and distribute normally.

Ex. $(4 + 5i)(2 + 3i)$

$= 8 + 12i + 10i + 15i^2$

$= 8 + 22i – 15$

$= -7 + 22i$

Practice

Compute the following: $(1 + 2i)(6 - 4i)$

Euler's formula

$$e^{ix} = \cos(x) + isin(x)$$

When we let x = pi, this leads to the famous identity

$$e^{i\pi} = -1$$

Multiplication

We have $z = (r_1, \theta_1)$ and $w = (r_2, \theta_2)$.

Express zw in terms of $r_1, r_2, \theta_1, \theta_2$

Roots of unity

- We say that the solutions to the polynomial $x^n - 1 = 0$ are the nth roots of unity. By the fundamental theorem of algebra. There are n of them.
- The most important fact about them is that the nth roots of unity will form an n – gon in the complex plane with the trivial solution of x = 1 (y = 0).
- Roots of unity are extremely important in advanced mathematics, but at lower levels, it's simply a quick trick to do stuff.

Roots of unity examples

- Suppose we had the polynomial $x^3 - 1 = 0$. To find the zeros, we could recall the trivial solution of x = 1 and use synthetic division with the factor (x – 1) to obtain $1 + x + x^2$. From here, we could just use the quadratic formula.
- But this is slow, let's used what we've just learned …

Problem

Given that n is even, what is the sum of the x – coordinates of the nth roots of unity? What is the sum of the y – coordinates of the nth roots of unity?

Problem

Find the 4 4th roots of unity.

Problems

Find the roots of

$$1 + x + x^2 + x^3 + x^4 + x^5 = 0$$

de Moivre's Formula (cont.)

Say we have $z = (r, \theta) = (r, \theta + 2\pi k)$.

We then raise z to the nth power

Then $z = (r, \theta + 2\pi k)^{\frac{n}{m}} = \left(r^{\frac{n}{m}}, \frac{n\theta}{m} + \frac{2\pi kn}{m}\right)$.