

Polynomials

Newport Math Club

Warm up Problems:

1. 7 coins are placed on a table. Initially there are 4 heads and 3 tails facing upwards. You desire to separate this group of 7 coins into 2 groups (not of the same size) with each group having an equal number of tails. Obviously, this would only be possible if you're allowed to flip some of the coins, and you are. What is your strategy in accomplishing this? Oh yeah, you're blindfolded.
2. Generalize. What is your strategy when there are x number of heads and y number of tails facing upwards?

What is a polynomial?

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

with nonnegative integer n

Question. Which of the following are polynomials?

a. $f(x) = a^2 x^2 + b^2 x + c^2$

b. $f(x) = \frac{1}{9} (x^3 + 9x^2 + 9)$

c. $f(x) = (\sqrt{x} + x + x^2)^2$

d. $f(x) = \left(\frac{1}{x}\right)^3 + \left(\frac{1}{x}\right)^2 + \left(\frac{1}{x}\right) + 1$

Fundamental theorem of algebra

(simplified version)

- Every polynomial of degree n will have n roots counting both real and non real roots up to their multiplicity.

(We say a polynomial has degree n where n is the largest exponent of x .)

- Ex. $f(x) = x^5 + x^4 + x^3 + x^2 + x + 1$ has 5 total zeros.
 $f(x) = ax^3 + bx^2 + cx + d$ has 3 total zeros

Basics of polynomials

Given $f(x) = x^4 + 3x^3 + 2x^2 + x + 9$

How would you evaluate $f(0)$ in your head?

$f(1)$?

$f(2)$?

$f(3)$?

Basics of polynomials

- Clearly, evaluating polynomials term by term is inefficient and highly susceptible to error. Luckily, there's something called nested form.

- We can write $f(x) = x^4 + 3x^3 + 2x^2 + x + 9$ as

$$x(x(x(x + 3) + 2) + 1) + 9$$

$f(3)$ is now easy to evaluate

Horner's Method (aka synthetic division)

- Synthetic division is used to divide a polynomial by a linear factor—and *only* a linear factor. Typically, you would use synthetic division to factor a polynomial in order to find its zeroes.
- Suppose you need to find the zeroes of the polynomial $x^2 + 5x + 6$. This is easiest when you can write the polynomial as the product of linear factors: $(x - a)(x - b)$, where a and b are the values of x for which the expression equals zero.

Horner's Method (cont.)

- Using the polynomial $x^2 + 5x + 6$...
- Write out the coefficients and a test zero (your guess at what one of the zeroes might be).
- Carry down the first coefficient, multiply it by the test zero, and write the resulting product under the second coefficient.

$$\begin{array}{r|rrr} -2 & 1 & 5 & 6 \end{array}$$

$$\begin{array}{r|rrr} -2 & 1 & 5 & 6 \\ & \downarrow & & \\ & -2 & & \\ \hline & 1 & & \end{array}$$

Horner's Method (cont.)

- Add down the second column, then repeat the multiplication with the resulting sum.
- Adding down the last column gives you the remainder.
- The other terms in the bottom row are the coefficients of the quotient.
- So, the factors of the polynomial are $(x + 2)$ and $(x + 3)$, and the zeroes are $x = -2$ and $x = -3$.

$$\begin{array}{r|rrr} -2 & 1 & 5 & 6 \\ & & -2 & -6 \\ \hline & 1 & 3 & \end{array}$$

$$\begin{array}{r|rrr} -2 & 1 & 5 & 6 \\ & & -2 & -6 \\ \hline & 1 & 3 & 0 \end{array}$$

$$x + 3$$

Horner's Method (cont.)

- Notice the similarities between expressing a polynomial in nested form and evaluating a quotient with synthetic division.
- Question to ponder: Why does synthetic division work? (You're on your own for this one...)

Problem

- Find the quotient

$$\frac{x^3 + x^2 + 4x + 12}{x + 2}$$

Problem

- For what values of x is the rational function $f(x)$ equal to an integer?

$$f(x) = \frac{x^3 + 2x^2 + x - 3}{x + 3}$$

Problem (cont.)

- So $f(x) = \frac{x^3 + 2x^2 + x - 3}{x + 3} = x^2 - x + 4 - \frac{15}{x + 3}$
- A final note about polynomial division. The end behavior of a rational function is just the “whole part” of the quotient and you ignore the remainder. Why?

Vieta's Formulas

- Vieta's formulas relate the sum, pair-wise sum, 3-wise sum, 4-wise sum, ... , product of the roots of a polynomial to its coefficients.
- This is best illustrated by example. Consider the equation $f(x) = x^2 + 7x - 11$. A quick check shows that the equation cannot be factored with integers. If we wanted to find the sum and product of f 's roots, it would be a tedious job to use the quadratic equation, and then to add or multiply the roots. Luckily, there's an easier way...

Vieta's Formulas (cont.)

- Say we have some polynomial with roots j and k . We can write our polynomial as

$$f(x) = a(x-j)(x-k)$$

Expanding yields (you may check this for yourself)

$$f(x) = ax^2 - a(j+k)x + ajk$$

And we notice that the sum $j+k$ is the negative second coefficient over the first, the product jk is the third coefficient over the first.

Vieta's Formulas (cont.)

- Going back to $f(x) = x^2 + 7x - 11$, the sum of the roots is just -7 and the product of the roots is just -11 .

Problem

- Let r and s be the roots of the polynomial

$$f(x) = 2x^2 + kx - 3$$

Express the sum $r^2 + s^2$ in terms of k .

Problem

$P(x)$ is a polynomial with real coefficients.

When $P(x)$ is divided by $x - 1$, the remainder is 3. When $P(x)$ is divided by $x - 2$, the remainder is 5. Find the remainder when $P(x)$ is divided by $x^2 - 3x + 2$.

hint: write $P(x)$ as $q(x)h(x) + r(x)$ where h is what you're dividing by and r is the remainder.

Binomial Theorem

- Binomial Theorem describes the expansion of powers of a binomial. For example

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

- Or more generally

$$(x + y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \dots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n$$

Questions

- Find the coefficient of the x^3y^5 term in the expansion of $(2x + 3y)^8$
- Why does it make sense that $(x + y)^n$ has $n + 1$ number of terms?

Problems

- How many terms does $(x + y + z)^n$ have in its expansion?
- Find the last three digits of 9^{1011} .

What is a complex number?

- A number in the form of $z = a + bi$ with real a , b and $i = \sqrt{-1}$

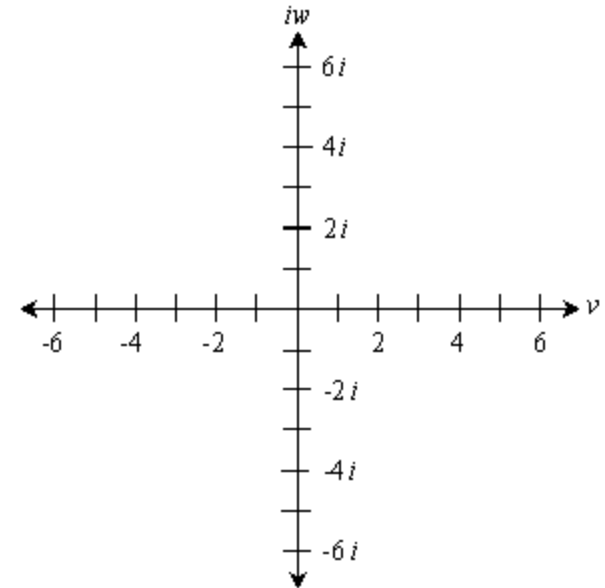
(Note: if $b \neq 0$ then z is complex. In addition, if $a = 0$, then z is “purely imaginary”)

- Consider the equation: $x^2 + 1 = 0$

Clearly, there are no real solutions, this is why we have complex numbers.

What is a complex number? (cont.)

- We represent numbers in the complex plane with the x axis representing the real part of the number, and the y axis representing the ~~fake~~ imaginary part of the number.
- The absolute value of a complex number, i.e., $|a + bi|$ is simply just the distance from the origin. This distance is $|a + bi| = \sqrt{a^2 + b^2}$. This comes from the Pythagorean Theorem.



Basics of complex numbers

- The conjugate of a complex number $z = a + bi$ is defined as $a - bi$. Simply flip the sign on the imaginary part of the complex number.
- Why do we care? The conjugate is used in simplifying quotients involving complex numbers (along with a variety of other uses)

Question

- Express $\frac{5 + 4i}{2 - 3i}$ as a single complex number.
- Suppose $z = a + bi$ is a complex number.
Real-ize the denominator of $\frac{1}{z}$.

Roots of unity

- We say that the solutions to the polynomial $x^n - 1 = 0$ are the n th roots of unity. By the fundamental theorem of algebra, there are n of them.
- The most important fact about them is that the n th roots of unity will form an n – gon in the complex plane with the trivial solution of $x = 1$ ($y = 0$).
- Roots of unity are extremely important in advanced mathematics, but at lower levels, it's simply a quick trick to do stuff.

Roots of unity examples

- Suppose we had the polynomial $x^3 - 1 = 0$. To find the zeros, we could recall the trivial solution of $x = 1$ and use synthetic division with the factor $(x - 1)$ to obtain $1 + x + x^2$. From here, we could just use the quadratic formula.
- But this is slow, let's use what we've just learned...

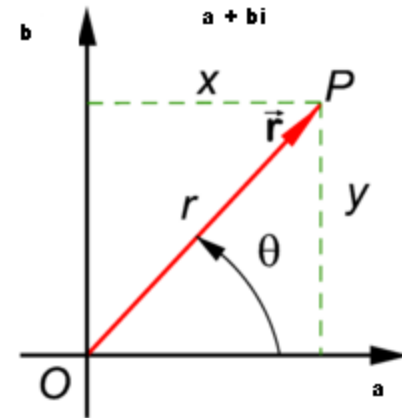
Problems

- What is the sum of the x – coordinates of the n th roots of unity? What is the sum of the y – coordinates of the n th roots of unity?
- Find the 4 4th roots of unity.

Basics of complex numbers

- It is often useful to write complex numbers in their polar representation. The polar representation of a point is expressed as (r, θ) .

where r is the distance from the origin and θ is the angle from the positive x axis. Given a and b , we know $r = \sqrt{a^2 + b^2}$. What is θ expressed in terms of a and b ?



Basics of complex numbers

$$\theta = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0 \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ 0 & \text{if } x = 0 \text{ and } y = 0 \end{cases}$$

Euler's formula

$$e^{ix} = \cos(x) + i\sin(x)$$

When we let $x = \pi$, this leads to the famous identity

$$e^{i\pi} = -1$$