Newport Math Club

NUMBER THEORY

Part 2

3 5 9 1 2 4 5 7 9 8 1 6 4 3 2 1 8 7 5 1 6 9 4

Number of Positive Integer Factors

- Now, what if we want to find how many factors a number has? Not *prime* factors – all of the positive integer factors
- Quick! List all of the positive integer factors of 36 and count them

  Answer: 9 (1, 2, 3, 4, 6, 9, 12, 18, 36)
- Quick! List all of the positive integer factors of 901800900 and count them

  Answer: 729 (No, I'm not listing them)

Number of Positive Integer Factors

- Clearly, there must be a faster way
- Luckily, we have a formula!
- Let us consider the number 4200
- First, we need to find the prime factorization
  - Answer: $2^3 \cdot 3 \cdot 5^2 \cdot 7$

Number of Positive Integer Factors

- Once we have listed the prime factors in this form (with multiplicities in exponents), we increment each exponent and multiply

$$4200 = 2^3 \cdot 3 \cdot 5^2 \cdot 7$$

$$(3 + 1)(1 + 1)(2 + 1)(1 + 1) =$$

$$4 \cdot 2 \cdot 3 \cdot 2 = 48$$

- The formula works by multiplying together the number of options for the multiplicity of each prime factor, but we add one to include 0

What did I do wrong?

- I found that the prime factorization of 420 was $2^2 \cdot 3 \cdot 5 \cdot 7$
- I then concluded that 420 has the following number of positive integer factors:

$$(2 + 1)(0 + 1)(0 + 1)(0 + 1) = 3 \cdot 1 \cdot 1 \cdot 1 = 3$$

- Clearly this isn't correct…

  What did I do wrong?

Your Turn

- Count the positive integer factors of 4590 without listing them all out

- Prime Factorization: $2 \cdot 3^3 \cdot 5 \cdot 17$
- Number of Factors: $2 \cdot 4 \cdot 2 \cdot 2 = 32$

Sum of Positive Integer Factors

- Now, what if we want to add up all the positive integer factors of a number?
- Quick! Add up all the positive integer factors of 12
  - Answer: 28
- Quick! Add up all the positive integer factors of 3072
  - Answer: 8188 (No, you don't want to do that)

Sum of Positive Integer Factors

- Once again, there is a faster way using a formula so you don't have to go crazy
- Let's consider the number 360
- Find the prime factorization
  - Answer: $2^3 \cdot 3^2 \cdot 5$

Sum of Positive Integer Factors

- $360 = 2^3 \cdot 3^2 \cdot 5$
- Create fractions with the incremented exponents, and decrement the top and bottom

$$\frac{2^{3+1}-1}{2-1} \cdot \frac{3^{2+1}-1}{3-1} \cdot \frac{5^{1+1}-1}{5-1} =$$

$$\frac{15}{1} \cdot \frac{26}{2} \cdot \frac{24}{4} = 15 \cdot 13 \cdot 6 = 1170$$

Sum of Positive Integer Factors

Another Example:

- Add up the positive integer factors of 1500
- Prime factorization: $2^2 \cdot 3 \cdot 5^3$
- Sum of positive integral factors:

$$\frac{2^{2+1}-1}{2-1}\cdot\frac{3^{1+1}-1}{3-1}\cdot\frac{5^{3+1}-1}{5-1}=$$

$$\frac{7}{1}\cdot\frac{8}{2}\cdot\frac{624}{4}=7\cdot 4\cdot 156=5460$$

Your Turn

- Add up the positive integer factors of 768
  - Answer: 2048: 512 · 4 ($768 = 2^8 \cdot 3$)
- Sum the positive integer factors of 950
  - Answer: 1860: 3 · 31 · 20 ($950 = 2 \cdot 5^2 \cdot 19$)
- Add up all of the integer factors of 5005
  - Answer: 0 (Did I trick you??)

- Now that we've used the formula a bit, why does it work?

Factorials

- The factorial operator, denoted by an exclamation mark (!), is a unary operator that acts on a single positive integer input
- To compute a factorial, you multiply the value by each integer below it down to 1
  - $1! = 1$
  - $2! = 2 \cdot 1 = 2$
  - $3! = 3 \cdot 2 \cdot 1 = 6$
  - $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$
- Note: 0! is defined as 1. Just remember that.

Factorials

- Factorials are very useful in number theory and combinatorics, which we will be discussing in a few weeks
- An interesting property of factorials is that they are recursive, meaning that
  - $1! = 1$
  - $2! = 2 \cdot 1! = 2 \cdot 1 \quad = 2$
  - $3! = 3 \cdot 2! = 3 \cdot 2 \quad = 6$
  - $4! = 4 \cdot 3! = 4 \cdot 6 \quad = 24$
  - $5! = 5 \cdot 4! = 5 \cdot 24 \quad = 120$

Factorials !

- When dividing a factorial by a number or another factorial, factor the numbers if necessary and cancel before you multiply

$$\frac{5!}{30} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 3 \cdot 5} = 1 \cdot 4 = 4$$

$$\frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 6 \cdot 5 \cdot 4 = 120$$

Your Turn

- Compute 7!
  - Answer: 5040
- Compute 10! (given that 9! = 362880)
  - Answer: 3628800
- What number do you have to multiply (7!) by to get (9!)?
  - Answer: 72
- Why don’t we include 0 in the factorials?
  - Answer: Every factorial would equal 0

How Many Zeros?

- Sometimes, we want to know how many zeros are at the end of a number
- We can learn something about this using number theory
- What is the prime factorization of 10?
- What is the prime factorization of 100?
- For every pair of prime factors 2 and 5, there will be another zero at the end of the number
  - Example: $2 \cdot 2 \cdot 5 \cdot 5 \cdot 5 \cdot 7$ ends in two zeros

How Many Zeros?

- For a factorial, there are ALWAYS more than enough 2s to pair with each 5, so we can count the multiplicity on 5 to determine the number of zeroes
- Example: Find the zeroes at the end of 31!
  - 5, 10, 15, 20, and 30 all have one 5
  - 25 has two 5s
  - Therefore, 31! ends in 7 zeros.

Your Turn

- How many zeroes does this number end in?
  $2^{99} \cdot 3^{100} \cdot 5^{125} \cdot 7^{76}$
  - Answer: 99
- How many zeroes does 128! end in?
  - Answer: 31
- How many zeroes does (52! / 24!) end in?
  - Answer: 8

QUESTIONS?

- Think back to what we've talked about

- Anything you want me to go back over?
- Anything you don't understand?

Challenge Question

- No talking – correct responses may earn an extra pi point. READY?
- If x is a positive integer with exactly 4 prime factors, each distinct, find the ratio of the number of positive integer factors of x^2 to the number of positive integer factors of x. Express your answer as a ratio in a:b form.

Answer: 81:16